The Energy of Conjugacy Class Graph of Some Order of Alternating Groups

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Abstract:

The energy of a graph $\boldsymbol{\Gamma}$, is the sum of all absolute values of the eigenvalues of the adjacency matrix which is indicated by $\boldsymbol{\varepsilon}(\boldsymbol{\Gamma})$. An adjacency matrix is a square matrix used to represent of finite graph where the rows and columns consist of 0 or 1-entry depending on the adjacency of the vertices of the graph. The group of even permutations of a finite set is known as an alternating group $\boldsymbol{A_n}$. The conjugacy class graph is a graph whose vertices are non-central conjugacy classes of a group $\boldsymbol{H}$ where two vertices are connected if their cardinalities are not coprime. In this paper, the conjugacy class of alternating group $\boldsymbol{A_n}$ of some order for $\boldsymbol{n \leq 10}$ and their energy are computed. The Maple2019 software and Groups, Algorithms, and Programming (GAP) are assisted for computations.

Keywords: Alternating Group, Conjugacy Class, Conjugacy Class Graph, Energy of Graph.

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1. Introduction

In this paper, A_n is an alternating group of order $n!/2$, and Γ indicated a simple graph. A conjugacy class is an equivalence relation, where the group is subdivided into disjoint sets. Suppose H is a finite group. Two elements x and y of H are called conjugate if there exists an element $h \in H$ with $hxh^{-1} = y$. The conjugacy is an equivalence relation; therefore, partition H into some equivalence classes. This means that every element of the group belongs to accurately one conjugacy class. The equivalence class that contains the element $x \in H$ is $x^H = \{hxh^{-1} : h \in H\}$ and is called the conjugacy class of x. The classes x^H and y^H are equal if and only if x and y are conjugate, and disjoint otherwise. The class number of H is the number of distinct (non-equivalent) conjugacy classes and we indicate it by $K(H)$. The elements of any group possibly subdivided into conjugacy classes.

The Konigsberg Bridge was the first problem of graph theory which was found by Elure in 1936. Euler solved this problem by drawing a graph with points and lines. Later, the utility of graph theory has been proven to many of innovation fields. A new graph was proposed in 1990 by Bertram *et.al*., called the conjugacy class graph, here we indicate as Γ_H^c (Bertram, et al., 1990). In 2013, a new graph has been found by Erfanian and Tolue , called the conjugate graph, here we indicate as Γ_H^{cl} (Erfanian & Tolue, 2012).

According to Woods (Woods, 2013), the study on the energy of general simple graphs was first defined by Gutman in 1978 (Gutman, 1978) inspired from the Huckel Molecular Orbital Theory proposed by Huckel in 1930. Huckel Molecular Orbital Theory has been used by chemists in approximating the energies related to $\pi-$electron orbitals in conjugated hydrocarbon. Later in 1956, Gunthard and Primas realized that the Huckel method is actually using the first degree polynomial of the adjacency matrix of a certain graph.

There are some researchers that have studied on the energy of graphs. For example, in 2004, Zhou and Balakrishnan (Zhou, 2004) and (Balakrishnan, 2004) studied the characteristics of energy of graphs while Bapat and Pati proved that the energy of a graph is never an odd integer in their research (Bapat & Pati, 2004). In the same year, Yu *et al*. (Yu, et al., 2004) dealt with the new upper bounds for the energy of graphs. In 2008, Pirzada and Gutman proved that the properties that the energy of a graph is never the square root of an odd integer (Prizada & Gutman, 2008).

Definition 1.1 (Goodman, 2003)

Let x and y be two elements in a finite group H, then x and y are called conjugate if there exists an element h in H such that $hxh^{-1} = y$.

Definition 1.2 (Goodman, 2003)

Let $x \in H$. The conjugacy class of x is the set $cl\,(x) = \{\, axa^{-1} \mid a \in H\}$.
In this paper, the notation $K(H)$ is used for the number of conjugacy classes in H, while $Z(H)$ is used for the center group H.

Proposition 1.1 (Fraleigh, 2002)

The conjugacy class of the identity element is its own class, namely $cl(e) = \{e\}$.

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Proposition 1.2 (Fraleigh, 2002)

Let x and y be the two elements in a finite group H. The elements x and y are conjugate if they belong in one conjugacy class, that is $cl(x) = cl(y)$ are the same.

Definition 1.3 (Bertram, et al., 1990)

Let H be a finite group and let $Z(H)$ be the center of H. The vertices of conjugacy class graph are non-central conjugacy classes of H i.e $|V(\Gamma H)| = K(H) - |Z(H)|$, where $K(H)$ is the number of conjugacy classes in H. Two vertices are adjacent if their cardinalities are not coprime (have common factor).

Definition 1.4 (Godsil & Royle, 2001)

A complete graph is a graph where each ordered pair of distinct vertices are adjacent, denoted by K_n. The graph Γ is a null graph, if it has no vertices. The graph is called empty if there are no edges between its vertices.

Definition 1.5 (Brouwer & Haemers, 2011)

The spectrum $S_P(\Gamma)$ of a graph Γ is defined as the eigenvalues of its adjacency matrix, that is, another matrix of two rows, the first row consists of the eigenvalues of the graph Γ and the second row consists of the multiplicities of the corresponding eigenvalues. That is if the distinct eigenvalues of Γ are $\lambda_1, \lambda_2, \lambda_3, \ldots., \lambda_k$ and their multiplicities are $m(\lambda_1), m(\lambda_2), m(\lambda_3), \ldots., m(\lambda_k)$ respectively, then we write

$$S_p(\Gamma) = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \ldots.\lambda_k \\ m(\lambda_1) & m(\lambda_2) & m(\lambda_3) & \ldots m(\lambda_k) \end{bmatrix}$$

2. Preliminaries

In this section, the conjugacy class of alternating group A_n are determined for $n \leq 10$ by using Groups, Algorithms and Programming (GAP).

Proposition 2.1: Let A_3 be a alternating group of order ${}^{3!}/_{2}$, $A_3 \cong \langle a : a^3 = e \rangle$. Then, the number of conjugacy class of A_3, $K(A_3) = 3$.

Proposition 2.2: Let A_4 be an alternating group of order ${}^{4!}/_{2}$, $A_4 \cong \langle a, b : a^3, b^3, (ab)^2 \rangle$. Then, the number of conjugacy classes of A_4, $K(A_4) = 4$.

Proposition 2.3: Let A_5 be an alternating group of order ${}^{5!}/_{2}$, $A_5 \cong \langle a, b, c : a^3, b^3, c^3, (ab)^2, (ac)^2, (bc)^2 \rangle$. Then, the number of conjugacy class of A_5, $K(A_5) = 5$.

Proposition 2.4: Let A_6 be an alternating group of order ${}^{6!}/_{2}$, $A_6 \cong \langle a, b : a^2, b^4, (ab)^5, (ab^2)^5 \rangle$. Then, the number of conjugacy class of A_6, $K(A_6) = 7$.

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Proposition 2.5: Let A_7 be an alternating group of order $^{7!}/_{2}$, $A_7 \cong \langle a, b: a^7, b^4, (ab^2)^3, (a^3b)^3, (aba^2b^3)^2 \rangle$. Then, the number of conjugacy class of A_7, $K(A_7) = 9$.

Proposition 2.6: Let A_8 be an alternating group of order $^{8!}/_{2}$. Then, the number of conjugacy class of A_8, $K(A_8) = 14$.

Proposition 2.7: Let A_9 be an alternating group of order $^{9!}/_{2}$. Then, the number of conjugacy class of A_9, $K(A_9) = 18$.

Proposition 2.8: Let A_{10} be an alternating group of order $^{10!}/_{2}$. Then, the number of conjugacy class of A_{10}, $K(A_{10}) = 24$.

3. Result and Discussion

First, we found the conjugacy class graph of some alternating groups A_n for $n \leq 10$. Then, the energy of these graphs is determined.

3.1 The Conjugacy Classes Graph of Alternating Groups A_n for $n \leq 10$

In the following, the conjugacy class graph of Alternating groups A_n, for $n \leq 10$ are determined. The results on the sizes of conjugacy classes of groups are used to get the conjugacy class graph. We start by finding the conjugacy class graph of A_4, since $H_1 = A_3$ is an abelian group, then $Z(A_3) = A_3$ has not non-central elements, then A_3 has no graph.

Theorem 3.1.1: Let $H_2 = A_4$ be an alternating group of order $^{4!}/_{2}$. Then, the conjugacy class graph of H_2, $\Gamma_{H_2}^{cl} = K_2 \cup K_1$.

Proof: Based on proposition 2.2, the number of conjugacy class of H_2 is 4 and $Z(H_2) = \{e\}$. Then, the number of non-central conjugacy classes of H_2 is equal to three. Therefore, the number of vertices in $\Gamma_{H_2}^{cl}$ is equal to three. Based on vertices adjacency of conjugacy class graph, thus $\Gamma_{H_2}^{cl}$ consists of one complete graph of K_2 with an isolated vertices, namely $cl(ab)$.

Theorem 3.1.2: Let $H_3 = A_5$ be an alternating group of order $^{5!}/_{2}$. Then, the conjugacy class graph of H_3, $\Gamma_{H_3}^{cl} = K_4$

Proof: Based on proposition 2.3, the number of conjugacy class of H_3 is 5 and $Z(H_3) = \{e\}$. Then, the number of non-central conjugacy classes of H_3 is equal to four. Therefore, the number of vertices in $\Gamma_{H_3}^{cl}$ is equal to four. Based on vertices adjacency of conjugacy class graph, thus $\Gamma_{H_3}^{cl}$ consists of a complete graph K_4.

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Theorem 3.1.3: Let $H_4 = A_6$ be an alternating group of order $^{6!}/_{2}$. Then, the conjugacy class graph of H_4, $\Gamma_{H_4}^{cl} = K_6$.

Proof: Based on proposition 2.4, the number of conjugacy class of H_4 is 7 and $Z(H_4) = \{e\}$. Then, the number of non-central conjugacy classes of H_4 is equal to six. Therefore, the number of vertices in $\Gamma_{H_4}^{cl}$ is equal to six. Based on vertices adjacency of conjugacy class graph, thus $\Gamma_{H_4}^{cl}$ consists of a complete graph K_6.

Theorem 3.1.4: Let $H_5 = A_7$ be an alternating group of order $^{7!}/_{2}$. Then, the conjugacy class graph of H_5, $\Gamma_{H_5}^{cl} = K_8$.

Proof: Based on proposition 2.5, the number of conjugacy class of H_5 is 9 and $Z(H_5) = \{e\}$. Then, the number of non-central conjugacy classes of H_5 is equal to eight. Therefore, the number of vertices in $\Gamma_{H_5}^{cl}$ is equal to eight. Based on vertices adjacency of conjugacy class graph, thus $\Gamma_{H_5}^{cl}$ consists of one complete graph K_8.

Theorem 3.1.5: Let $H_6 = A_8$ be an alternating group of order $^{8!}/_{2}$. Then, the conjugacy class graph of H_6, $\Gamma_{H_6}^{cl} = K_{13}$.

Proof: Based on proposition 2.6, the number of conjugacy class of H_5 is 14 and $Z(H_6) = \{e\}$. Then, the number of non-central conjugacy classes of H_6 is equal to 13. Therefore, the number of vertices in $\Gamma_{H_6}^{cl}$ is equal to 13. Based on vertices adjacency of conjugacy class graph, thus $\Gamma_{H_6}^{cl}$ consist of one complete components of K_{13}.

Theorem 3.1.6: Let $H_7 = A_9$ be an alternating group of order $^{9!}/_{2}$. Then, the conjugacy class graph of H_7, $\Gamma_{H_7}^{cl} = K_{17}$.

Proof: Based on proposition 2.7, the number of conjugacy class of H_7 is 18 and $Z(H_7) = \{e\}$. Then, the number of non-central conjugacy classes of H_7 is equal to 17. Therefore, the number of vertices in $\Gamma_{H_7}^{cl}$ is equal to 17. Based on vertices adjacency of conjugacy class graph, thus $\Gamma_{H_7}^{cl}$ consist of one complete components of K_{17}.

Theorem 3.1.7: Let $H_8 = A_{10}$ be an alternating group of order $^{10!}/_{2}$. Then, the conjugacy class graph of H_8, $\Gamma_{H_8}^{cl} = K_{23}$.

Proof: Based on proposition 2.8, the number of conjugacy class of H_8 is 24 and $Z(H_8) = \{e\}$. Then, the number of non-central conjugacy classes of H_8 is equal to 23. Therefore, the number of vertices in $\Gamma_{H_8}^{cl}$ is equal to H_8. Based on vertices adjacency of conjugacy class graph, thus $\Gamma_{H_8}^{cl}$ consists of one complete components of K_{23}.

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3.2 Energy of Conjugacy Class Graph of Alternating Groups A_n, for $n \leq 10$

In this section, the energy of conjugacy class graph of some alternating groups A_n, for $n \leq 10$ are determined, the following are the result:

Theorem 3.2.1: The energy of the conjugacy class of $H_2 = A_4$ is $\varepsilon\left(\Gamma_{H_2}^{cl}\right)$ =2.

Proof: Suppose $H_2 = A_4$. Then, the adjacency matrix B for the conjugacy class graph $\Gamma_{H_2}^{cl} = K_2 \cup K_1$ is given in the following:

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the characteristic polynomial of B is given as the following:

$$P(\lambda) = \lambda^3 - \lambda.$$

Hence, the spectrum of the conjugacy class graph for the group H_2 can be written as:

$$S_P(\Gamma_{H_2}^{cl}) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore, the energy of the conjugacy class graph for A_4 is $\varepsilon\left(\Gamma_{H_2}^{cl}\right) = \sum_{i=1}^{n} |\lambda_i|$ =2.

Theorem 3.2.2: Let $H_3 = A_5$. Then, the energy of the conjugacy class of H_3, $\varepsilon\left(\Gamma_{H_3}^{cl}\right)$ =6.

Proof: Let $H_3 = A_5$. Then, the adjacency matrix B for the conjugacy class graph $\Gamma_{H_3}^{cl} = K_4$ is given in the following:

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Thus, the characteristic polynomial of B is given as the following:

$$P(\lambda) = (\lambda + 1)^3.(\lambda - 3)$$

Hence, the spectrum of the conjugacy class graph for the group H_3 can be written as:

$$S_P(\Gamma_{H_3}^{cl}) = \begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}$$

Therefore, the energy of the conjugacy class graph for A_5 is $\varepsilon\left(\Gamma_{H_3}^{cl}\right) = \sum_{i=1}^{n} |\lambda_i|$ =6.

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Theorem 3.2.3: Let $H_4 = A_6$. Then, the energy of the conjugacy class of H_4, $\varepsilon\left(\Gamma_{H_4}^{cl}\right) = 10$.

Proof: Let $H_4 = A_6$. Then, the adjacency matrix B for the conjugacy class graph $\Gamma_{H_4}^{cl} = K_6$ is given in the following:

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Thus, the characteristic polynomial of B is given as the following:

$$P(\lambda) = (\lambda + 1)^5 . (\lambda - 5)$$

Hence, the spectrum of the conjugacy class graph for the group H_4 can be written as:

$$S_P(\Gamma_{H_4}^{cl}) = \begin{bmatrix} -1 & 5 \\ 5 & 1 \end{bmatrix}$$

Therefore, the energy of the conjugacy class graph for A_6 is $\varepsilon\left(\Gamma_{H_4}^{cl}\right) = \sum_{i=1}^{n} |\lambda_i| = 10$.

Theorem 3.2.4: Let $H_5 = A_7$. Then, the energy of the conjugacy class of H_5, $\varepsilon\left(\Gamma_{H_5}^{cl}\right) = 14$.

Proof: Suppose $H_5 = A_7$. Then, the adjacency matrix A for the conjugacy class graph $\Gamma_{H_5}^{cl} = K_8$ is given in the following:

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Thus, the characteristic polynomial of B is given as the following:

$$P(\lambda) = (\lambda + 1)^7 . (\lambda - 7)^1$$

Hence, the spectrum of the conjugacy class graph for the group H_5 can be written as:

$$S_P(\Gamma_{H_5}^{cl}) = \begin{bmatrix} -1 & 7 \\ 7 & 1 \end{bmatrix}$$

Therefore, the energy of the conjugacy class graph for A_7 is $\varepsilon\left(\Gamma_{H_5}^{cl}\right) = \sum_{i=1}^{n} |\lambda_i| = 14$.

Theorem 3.2.5: Let $H_6 = A_8$. Then, the energy of the conjugacy class of H_6, $\varepsilon\left(\Gamma_{H_6}^{cl}\right) = 24$.

Proof: Suppose $H_6 = A_8$. Then, the adjacency matrix B for the conjugacy class graph $\Gamma_{H_6}^{cl} = K_{13}$ is given in the following:

$$B = J_{13\times 13} - I_{13\times 13}$$

Thus, the characteristic polynomial of B is given as the following:

$$P(\lambda) = (\lambda + 1)^{12} (\lambda - 12)$$

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Hence, the spectrum of the conjugacy class graph for the group H_6 can be written as:

$$S_P(\Gamma_{H_6}^{cl}) = \begin{bmatrix} -1 & 12 \\ 12 & 1 \end{bmatrix}$$

Therefore, the energy of the conjugacy class graph for A_8 is $\varepsilon(\Gamma_{H_6}^{cl}) = \sum_{i=1}^{n}|\lambda_i|$ =24.

Theorem 3.2.6: Let $H_7 = A_9$. Then, the energy of the conjugacy class of H_7, $\varepsilon(\Gamma_{H_7}^{cl}) = 32$.

Proof: Suppose $H_7 = A_9$. Then, the adjacency matrix B for the conjugacy class graph $\Gamma_{H_7}^{cl} = K_{17}$ is given in the following:

$$B = J_{17\times17} - I_{17\times17}$$

Thus, the characteristic polynomial of B is given as the following:

$$P(\lambda) = (\lambda + 1)^{16}(\lambda - 16)$$

Hence, the spectrum of the conjugacy class graph for the group H_7 can be written as:

$$S_P(\Gamma_{H_7}^{cl}) = \begin{bmatrix} -1 & 16 \\ 16 & 1 \end{bmatrix}$$

Therefore, the energy of the conjugacy class graph for A_9 is $\varepsilon(\Gamma_{H_7}^{cl}) = \sum_{i=1}^{n}|\lambda_i|$ =32.

Theorem 3.2.7: Let $H_8 = A_{10}$. Then, the energy of the conjugacy class of H_8, $\varepsilon(\Gamma_{H_8}^{cl}) = 44$.

Proof: Suppose $H_8 = A_{10}$. Then, the adjacency matrix B for the conjugacy class graph $\Gamma_{H_8}^{cl} = K_{23}$is given in the following:

$$B = J_{23\times23} - I_{23\times23}$$

Thus, the characteristic polynomial of B is given as the following:

$$P(\lambda) = (\lambda + 1)^{22}.(\lambda - 22)$$

Hence, the spectrum of the conjugacy class graph for the group H_8 can be written as:

$$S_P(\Gamma_{H_8}^{cl}) = \begin{bmatrix} -1 & 22 \\ 22 & 1 \end{bmatrix}$$

Therefore, the energy of the conjugacy class graph for A_{10} is $\varepsilon(\Gamma_{H_8}^{cl}) = \sum_{i=1}^{n}|\lambda_i| = 44$.

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4. Conclusion

In this paper, first, the conjugacy classes of alternating group A_n, when $n \leq 10$ are determined by using GAP programming. Next, the conjugacy class graphs are determined. Finally, the energies of conjugacy classes of alternating group A_n, for $n \leq 10$ are determined. The results are summarized in the table below:

Table 1- Energy of Conjugacy Class Graph of Some Alternating Groups

No	Groups	Order	No. Conjugacy Classes	Conjugacy Class Graph	Energy
1	$H_2 = A_4$	$4!/2 = 12$	4	K_2	2
2	$H_3 = A_5$	$5!/2 = 60$	5	K_4	6
3	$H_4 = A_6$	$6!/2 = 360$	7	K_6	10
4	$H_5 = A_7$	$7!/2 = 2520$	9	K_8	14
5	$H_6 = A_8$	$8!/2 = 20160$	14	K_{13}	24
6	$H_7 = A_9$	$9!/2 = 181440$	18	K_{17}	32
7	$H_8 = A_{10}$	$10!/2 = 1814400$	24	K_{23}	44

Note: This Research Paper has Already Been Published With Scientific Inaccuracy and has Been Republished After Editing and Removing The Error.

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